

# SPECIAL FUNCTIONS AND TWISTED $L$ -SERIES

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ABSTRACT. We introduce a generalization of the Anderson-Thakur special function, and we prove a rationality result for several variable twisted  $L$ -series associated to shtuka functions.

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## 1. INTRODUCTION

Let  $X = \mathbb{P}^1/\mathbb{F}_q$  be the projective line over a finite field  $\mathbb{F}_q$  having  $q$  elements and let  $K$  be its function field. Let  $\infty$  be a closed point of  $X$  of degree  $d_\infty = 1$ . Then  $K = \mathbb{F}_q(\theta)$  for some  $\theta \in K$  such that  $\theta$  has a pole of order one at  $\infty$ . We set  $A = \mathbb{F}_q[\theta]$ . Following Anderson ([2], see also [22]), we consider:

$$Y = K \otimes_{\mathbb{F}_q} X.$$

Let  $\mathbb{K} = \text{Frac}(K \otimes_{\mathbb{F}_q} K)$  be the function field of  $Y$ . We identify  $K$  with  $K \otimes 1 \subset \mathbb{K}$ . If we set  $t = 1 \otimes \theta$ , then  $\mathbb{K} = K(t)$ . Let  $\tau : \mathbb{K} \rightarrow \mathbb{K}$  be the homomorphism of  $\mathbb{F}_q(t)$ -algebras such that:

$$\forall x \in K, \quad \tau(x) = x^q.$$

Let  $\infty \in Y(K)$  be the pole of  $t$ , and let  $\xi \in Y(K)$  be the point corresponding to the kernel of the homomorphism of  $K$ -algebras  $K \otimes_{\mathbb{F}_q} K \rightarrow K$  which sends  $t$  to  $\theta$ .

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Then the divisor of  $f := t - \theta$  is equal to  $(\xi) - (\infty)$ . The function  $t - \theta$  is a shtuka function, and in particular:

$$\forall a \in A, \quad a(t) = \sum_{k=0}^{\deg_{\theta} a} C_{a,i} f \cdots f^{(i-1)}, \text{ with } C_{a,i} \in A.$$

The map  $C : A \rightarrow A\{\tau\}, a \mapsto C_a := \sum_{k=0}^{\deg_{\theta} a} C_{a,i} \tau^i$  is a homomorphism of  $\mathbb{F}_q$ -algebras called the Carlitz module. Note that:

$$C_{\theta} = \theta + \tau.$$

There exists a unique element  $\exp_C \in K\{\{\tau\}\}$  such that  $\exp_C \equiv 1 \pmod{\tau}$  and:

$$\forall a \in A, \quad \exp_C a = C_a \exp_C.$$

Let  $\mathbb{C}_{\infty}$  be the completion of a fixed algebraic closure of  $K_{\infty} := \mathbb{F}_q((\frac{1}{\theta}))$ . Then  $\exp_C$  defines an entire function on  $\mathbb{C}_{\infty}$ , and:

$$\text{Ker } \exp_C = \tilde{\pi}A,$$

for some  $\tilde{\pi} \in \mathbb{C}_{\infty}^{\times}$  (well-defined modulo  $\mathbb{F}_q^{\times}$ ) called the Carlitz period. We consider  $\mathbb{T}$  the Tate algebra in the variable  $t$  with coefficients in  $\mathbb{C}_{\infty}$ , i.e.  $\mathbb{T} := \mathbb{C}_{\infty} \widehat{\otimes}_{\mathbb{F}_q} A$ . Let  $\tau : \mathbb{T} \rightarrow \mathbb{T}$  be the continuous homomorphism of  $\mathbb{F}_q[t]$ -algebras such that  $\forall x \in \mathbb{C}_{\infty}, \tau(x) = x^q$ . Anderson and Thakur ([3]) showed that:

$$\{x \in \mathbb{T}, \tau(x) = fx\} = \omega \mathbb{F}_q[t],$$

where  $\omega \in \mathbb{T}^{\times}$  is such that:

$$f\omega|_{\xi} = \tilde{\pi}.$$

The function  $\omega$  is called the Anderson-Thakur special function attached to the Carlitz module  $C$ . This function is intimately connected to Thakur-Gauss sums ([7]).

In 2012, Pellarin ([19]) initiated the study of a twist of the Carlitz module by the shtuka function  $f$ . Let's consider the following homomorphism of  $\mathbb{F}_q$ -algebras  $\varphi : A \rightarrow A[t]\{\tau\}, \theta \mapsto \theta + f\tau$ . Then, one observes that  $C$  and  $\varphi$  are isomorphic over  $\mathbb{T}$ , i.e. we have the following equality in  $\mathbb{T}\{\tau\}$ :

$$\forall a \in A, \quad C_a \omega = \omega \varphi_a.$$

To such an object, one can associate the special value of some twisted  $L$ -function (see [8]):

$$\mathcal{L} = \sum_{a \in A, a \text{ monic}} \frac{a(t)}{a} \in \mathbb{T}^{\times}.$$

Then, using the Anderson log-algebraicity Theorem for the Carlitz module ([1], see also [18], [9]), Pellarin proved the following remarkable rationality result:

$$\frac{\mathcal{L}\omega}{\tilde{\pi}} = \frac{1}{f} \in \mathbb{K}.$$

This result has been extended to the case of “several variables” ([8], [12]) using methods developed by Taelman ([20], [21], [10], [14], [15], [13]). This kind of rationality results leads to new advances in the arithmetic of function fields (see [8], [11], [6]).

The aim of this paper is to extend the previous results to the general context, i.e. for any smooth projective geometrically irreducible curve  $X/\mathbb{F}_q$  of genus  $g$

and any closed point  $\infty$  of degree  $d_\infty$  of  $X$ . In particular, we obtain a rationality result similar to that of Pellarin (Theorem 5.6). Our result involves twisted  $L$ -series (see [5]) and a generalization of the Anderson-Thakur special function. The involved techniques are based on ideas developed in [6] where an analogue of Stark Conjectures is proved for sign-normalized rank one Drinfeld modules.

We should mention that Green and Papanikolas ([17]) have recently studied the particular case  $g = 1$  and  $d_\infty = 1$  and, in this case, they have obtained explicit formulas similar to that obtained by Pellarin (in the case  $g = 0$  and  $d_\infty = 1$ ).

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## 2. NOTATION AND BACKGROUND

### 2.1. Notation.

Let  $X/\mathbb{F}_q$  be a smooth projective geometrically irreducible curve of genus  $g$ , and  $\infty$  be a closed point of degree  $d_\infty$  of  $X$ . Denote by  $K$  the function field of  $X$ , and by  $A$  the ring of elements of  $K$  which are regular outside  $\infty$ . The completion  $K_\infty$  of  $K$  at the place  $\infty$  has residue field  $\mathbb{F}_\infty$ . We fix an algebraic closure  $\overline{K}_\infty$  of  $K_\infty$  and denote by  $\mathbb{C}_\infty$  the completion of  $\overline{K}_\infty$ .

We will fix a sign function  $\text{sgn} : K_\infty^\times \rightarrow \mathbb{F}_\infty^\times$  which is a group homomorphism such that  $\text{sgn}|_{\mathbb{F}_\infty^\times} = \text{Id}|_{\mathbb{F}_\infty^\times}$ . We fix  $\pi \in K \cap \text{Ker}(\text{sgn})$  and such that  $K_\infty = \mathbb{F}_\infty((\pi))$ . Let  $v_\infty : \mathbb{C}_\infty \rightarrow \mathbb{Q} \cup \{+\infty\}$  be the valuation on  $\mathbb{C}_\infty$  normalized such that  $v_\infty(\pi) = 1$ . Observe that:

$$\forall x \in K^\times, \quad \deg(xA) = -d_\infty v_\infty(x).$$

Let  $\overline{K}$  be the algebraic closure of  $K$  in  $\mathbb{C}_\infty$ .

Let  $\mathcal{I}(A)$  be the group of non-zero fractional ideals of  $A$ . We have a natural surjective group homomorphism  $\deg : \mathcal{I}(A) \rightarrow \mathbb{Z}$ , such that for  $I \in \mathcal{I}(A)$ ,  $I \subset A$ , we have:

$$\deg I = \dim_{\mathbb{F}_q} A/I.$$

Let  $\mathcal{P}(A) = \{xA, x \in K^\times\}$ , then  $\text{Pic}(A) = \frac{\mathcal{I}(A)}{\mathcal{P}(A)}$  is a finite abelian group.

Let  $I_K$  be the group of idèles of  $K$ , and  $H/K$  be the finite abelian extension of  $K$ ,  $H \subset \mathbb{C}_\infty$ , corresponding via class field theory to the following subgroup of  $I_K$ :

$$K^\times \ker \text{sgn} \prod_{v \neq \infty} O_v^\times,$$

where for a place  $v \neq \infty$  of  $K$ ,  $O_v^\times$  denotes the group of units of the  $v$ -adic completion of  $K$ . Then  $H/K$  is a finite extension of degree  $|\text{Pic}(A)| \mid \frac{q^{d_\infty}-1}{q-1}$ , unramified outside  $\infty$ , and the decomposition group of  $\infty$  in  $H/K$  is equal to its inertia group and is isomorphic to  $\frac{\mathbb{F}_\infty^\times}{\mathbb{F}_q^\times}$ . Set  $G = \text{Gal}(H/K)$ . If we define  $\mathcal{P}_+(A) = \{xA, x \in K^\times, \text{sgn}(x) = 1\}$ , then the Artin map

$$(\cdot, H/K) : \mathcal{I}(A) \longrightarrow G.$$

induces a group isomorphism:

$$\frac{\mathcal{I}(A)}{\mathcal{P}_+(A)} \simeq G.$$

For  $I \in \mathcal{I}(A)$ , we set:

$$\sigma_I = (I, H/K) \in G.$$

Let  $H_A$  be the Hilbert class field of  $A$ , i.e.  $H_A/K$  corresponds to the following subgroup of the idèles of  $K$ :

$$K^\times K_\infty^\times \prod_{v \neq \infty} O_v^\times.$$

Then  $H/H_A$  is totally ramified at the places of  $H_A$  above  $\infty$ . Furthermore:

$$\text{Gal}(H/H_A) \simeq \frac{\mathbb{F}_\infty^\times}{\mathbb{F}_q^\times}.$$

We denote by  $B$  the integral closure of  $A$  in  $H$  and  $B'$  the integral closure of  $A$  in  $H_A$ . Observe that  $\mathbb{F}_\infty \subset B$ .

## 2.2. Sign-normalized rank one Drinfeld modules.

We define the map  $\tau : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty, x \mapsto x^q$ . By definition, a sign-normalized rank one Drinfeld module is a homomorphism of  $\mathbb{F}_q$ -algebras  $\phi : A \rightarrow \mathbb{C}_\infty\{\tau\}$  such that there exists  $n(\phi) \in \{0, \dots, d_\infty - 1\}$  with the following property:

$$\forall a \in A, \quad \phi_a = a + \dots + \text{sgn}(a) q^{n(\phi)} \tau^{\deg a}.$$

Let  $n \in \{0, \dots, d_\infty - 1\}$ . We denote by  $\text{Drin}_n$  the set of sign-normalized rank one Drinfeld modules  $\phi$  with  $n(\phi) = n$ , and by  $\text{Drin} = \bigcup_{n=0}^{d_\infty-1} \text{Drin}_n$  the set of sign-normalized rank one Drinfeld modules. By [16], Corollary 7.2.17,  $\text{Drin}$  is a finite set and we have:

$$|\text{Drin}| = |\text{Pic}(A)| \frac{q^{d_\infty} - 1}{q - 1}.$$

Let  $\phi \in \text{Drin}$  be a sign-normalized rank one Drinfeld module, we say that  $\phi$  is standard if  $\text{Ker exp}_\phi$  is a free  $A$ -module, where  $\text{exp}_\phi : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$  is the exponential map attached to  $\phi$  (see for example [16], paragraph 4.6).

**Lemma 2.1.** *Let  $n \in \{0, \dots, d_\infty - 1\}$ . We have:*

$$|\text{Drin}_n| = \frac{1}{d_\infty} |\text{Pic}(A)| \frac{q^{d_\infty} - 1}{q - 1}.$$

*Let  $\phi$  in  $\text{Drin}_n$  and let  $[\phi]$  denote the set of the  $\phi'$  in  $\text{Drin}_n$  which are isomorphic to  $\phi$ . Then:*

$$\forall \phi \in \text{Drin}_n, \quad |[\phi]| = \frac{q^{d_\infty} - 1}{q - 1}.$$

*In particular, if  $[\text{Drin}_n] = \{[\phi], \phi \in \text{Drin}_n\}$ , we have:*

$$|[\text{Drin}_n]| = \frac{1}{d_\infty} |\text{Pic}(A)|.$$

*Proof.* Let  $\psi : A \rightarrow H\{\tau\}$  be a sign-normalized rank one Drinfeld module (see [16], chapter 7). Let  $n(\psi) \in \mathbb{Z}$  be such that:

$$\forall a \in A, \quad \psi_a = a + \dots + \text{sgn}(a) q^{n(\psi)} \tau^{\deg a}.$$

Then the set of sign-normalized rank one Drinfeld modules is exactly  $\text{Drin} = \{\psi^\sigma, \sigma \in G\}$ . Let  $\sigma \in G$  and write  $\sigma = (I, H/K)$  for some  $I \in \mathcal{I}(A)$ . We have:

$$\forall a \in A, \quad \psi_a^\sigma = a + \dots + \text{sgn}(a) q^{n(\psi) + \deg(I)} \tau^{\deg a}.$$

Note that  $\deg : \mathcal{I}(A) \rightarrow \mathbb{Z}$  induces a surjective homomorphism of finite abelian groups:

$$\deg : \frac{\mathcal{I}(A)}{\mathcal{P}_+(A)} \rightarrow \frac{\mathbb{Z}}{d_\infty \mathbb{Z}}.$$

Since there are exactly  $|\text{Pic}(A)| \frac{q^{d_\infty}-1}{q-1}$  sign-normalized rank one Drinfeld modules and  $d_\infty$  divides  $|\text{Pic}(A)|$ , we get the first assertion.

Let  $\phi \in \text{Drin}_n$  and let  $\phi' \in [\phi]$ . Then there exists  $\alpha \in \mathbb{C}_\infty^\times$  such that:

$$\forall a \in A, \quad \alpha \phi_a = \phi'_a \alpha.$$

Thus,  $\alpha \in \mathbb{F}_\infty^\times$ . Since  $\text{End}_{\mathbb{C}_\infty}(\phi) = \{\phi_a, a \in A\}$ , we obtain:

$$\text{End}_{\mathbb{C}_\infty}(\phi) \cap \mathbb{F}_\infty = \mathbb{F}_q.$$

Hence,

$$|[\phi]| = \frac{q^{d_\infty} - 1}{q - 1}.$$

□

**Lemma 2.2.** *There are exactly  $\frac{q^{d_\infty}-1}{q-1}$  standard elements in  $\text{Drin}$ . Furthermore, if  $\phi$  is such a Drinfeld module, then  $[\phi]$  is the set of standard elements in  $\text{Drin}$ .*

*Proof.* By [16], Corollary 4.9.5 and [16], Theorem 7.4.8, there exists  $\phi \in \text{Drin}$  such that  $\phi$  is standard. In particular,  $\text{Drin} = \{\phi^\sigma, \sigma \in G\}$ . Again, by [16], Corollary 4.9.5 and [16], Theorem 7.4.8, the Drinfeld module  $\phi^\sigma$  is standard if and only if  $\sigma|_{H_A} = \text{Id}_{H_A}$ . The Lemma follows. □

### 2.3. Shtuka functions.

Let  $\bar{X} = \mathbb{C}_\infty \otimes_{\mathbb{F}_q} X$ ,  $\bar{A} = \mathbb{C}_\infty \otimes_{\mathbb{F}_q} A$ , and let  $F$  be the function field of  $\bar{X}$ , i.e.  $F = \text{Frac}(\bar{A})$ . We will identify  $\mathbb{C}_\infty$  with its image  $\mathbb{C}_\infty \otimes 1$  in  $F$ . There are  $d_\infty$  points in  $\bar{X}(\mathbb{C}_\infty)$  above  $\infty$ , and we denote the set of such points by  $S_\infty$ . Observe that  $\bar{A}$  is the set of elements of  $F/\mathbb{C}_\infty$  which are “regular outside  $\infty$ ”. We denote by  $\tau : F \rightarrow F$  the homomorphism of  $K$ -algebras such that:

$$\tau|_{\bar{A}} = \tau \otimes 1.$$

For  $m \in \mathbb{Z}$ , we also set:

$$\forall x \in F, \quad x^{(m)} = \tau^m(x).$$

Let  $P$  be a point of  $\bar{X}(\mathbb{C}_\infty)$ . We denote by  $P^{(i)}$  the point of  $\bar{X}(\bar{K})$  obtained by applying  $\tau^i$  to the coordinates of  $P$ . If  $D = \sum_{j=1}^n n_{P_j} P_j \in \text{Div}(\bar{X})$ , with  $P_j \in \bar{X}(\mathbb{C}_\infty)$ , and  $n_{P_j} \in \mathbb{Z}$ , we set:

$$D^{(i)} = \sum_{j=1}^n n_{P_j} P_j^{(i)}.$$

If  $D = (x)$ ,  $x \in F^\times$ , then:

$$D^{(i)} = (x^{(i)}).$$

We consider  $\xi \in \bar{X}(\mathbb{C}_\infty)$  the point corresponding to the kernel of the map:

$$\bar{A} \rightarrow \mathbb{C}_\infty, \quad \sum_i x_i \otimes a_i \mapsto \sum_i x_i a_i.$$

Let  $\rho : K \rightarrow F, x \mapsto 1 \otimes x$  and set  $t = \rho(\pi^{-1})$ .

Let  $\bar{\infty} \in S_\infty$ . We identify the  $\bar{\infty}$ -adic completion of  $F$  to

$$\mathbb{C}_\infty((\frac{1}{t})).$$

Let  $\text{sgn}_\infty : \mathbb{C}_\infty((\frac{1}{t}))^\times \rightarrow \mathbb{C}_\infty^\times$  be the group homomorphism such that  $\text{Ker}(\text{sgn}_\infty) = t^\mathbb{Z} \times (1 + \frac{1}{t}\mathbb{C}_\infty[[\frac{1}{t}]])$ , and  $\text{sgn}_\infty|_{\mathbb{C}_\infty^\times} = \text{Id}|_{\mathbb{C}_\infty^\times}$ .

Let  $\phi \in \text{Drin}$ . For  $a \in A$ , we write  $\phi_a = \sum_{i=0}^{\deg a} \phi_{a,i} \tau^i$ ,  $\phi_{a,i} \in H$ . By [16], chapter 6, and [16], Proposition 7.11.4, there exists  $\bar{\infty} \in S_\infty$  and  $f_\phi \in F^\times$  such that:

$$\forall a \in A, \quad \rho(a) = \sum_{i=0}^{\deg a} \phi_{a,i} f_\phi \cdots f_\phi^{(i-1)},$$

and the divisor of  $f_\phi$  is of the form:

$$(f_\phi) = V^{(1)} - V + (\xi) - (\bar{\infty}),$$

where  $V$  is some effective divisor of degree  $g$ . Let  $(\infty) = \sum_{\bar{\infty}' \in S_\infty} (\bar{\infty}')$ . Set

$$W(\mathbb{C}_\infty) = \cup_{m \geq 0} L(V + m(\infty)),$$

and

$$L(V + m(\infty)) = \{x \in F^\times, (x) + V + m(\infty) \geq 0\} \cup \{0\}.$$

We have:

$$W(\mathbb{C}_\infty) = \oplus_{i \geq 0} \mathbb{C}_\infty f_\phi \cdots f_\phi^{(i-1)}.$$

The function  $f_\phi$  is called the shtuka function attached to  $\phi$ , and we say that  $\phi$  is the signed-normalized rank one Drinfeld module associated to  $f_\phi$ . We define the set of shtuka functions to be:

$$\text{Sht} = \{f_\phi, \phi \in \text{Drin}\}.$$

Then, the map  $\text{Drin} \rightarrow \text{Sht}, \phi \rightarrow f_\phi$  is a bijection called the Drinfeld correspondence.

**Remark 2.3.** There is a misprint in [16], page 229. In fact, as we will see in the proof of Lemma 3.3, when  $d_\infty > 1$ , we do not have:  $\text{sgn}_{\bar{\infty}(-1)}(f_\phi)^{\frac{q^{d_\infty}-1}{q-1}} = 1$  as stated in the *loc. cit.*

### 3. SPECIAL FUNCTIONS ATTACHED TO SHTUKA FUNCTIONS

#### 3.1. Basic properties of a shtuka function.

Let  $\mathbb{H} = \text{Frac}(H \otimes_{\mathbb{F}_q} A)$ , and  $\mathbb{K} = \text{Frac}(K \otimes_{\mathbb{F}_q} A)$ . Recall that  $G = \text{Gal}(H/K)$  and we will identify  $G$  with the Galois group of  $\mathbb{H}/\mathbb{K}$ . Let  $f \in \text{Sht}$ , and let  $\phi \in \text{Drin}_{n(\phi)}$  be the sign-normalized rank one Drinfeld module attached to  $f$  for some  $n(\phi) \in \{0, \dots, d_\infty - 1\}$ . Then  $\phi : A \rightarrow B\{\tau\}$  is a homomorphism of  $\mathbb{F}_q$ -algebras such that:

$$\forall a \in A, \quad \phi_a = \sum_{i=0}^{\deg a} \phi_{a,i} \tau^i,$$

where  $\phi_{a,0} = a$ ,  $\phi_{a,\deg a} = \text{sgn}(a)^{q^{n(\phi)}}$ , and  $\rho(a) = \sum_{i=0}^{\deg a} \phi_{a,i} f \cdots f^{(i-1)}$ . Recall that there exists an effective  $\mathbb{H}$ -divisor  $V$  ([16], chapter 6) of degree  $g$  such that the divisor of  $f$  is:

$$(f) = V^{(1)} - V + (\xi) - (\bar{\infty}),$$

for some  $\infty \in S_\infty$ . By [16], Lemma 7.11.3,  $\xi, \infty^{(-1)}$  do not belong to the support of  $V$ . Let  $v_\infty$  be the normalized valuation on  $\mathbb{H}$  attached to  $\infty$  ( $v_\infty(t) = -1$ ). Note that  $v_\infty(f) \leq -1$  and, when  $d_\infty > 1$ ,  $\infty$  can a priori belong to the support of  $V$ . We identify the  $\infty$ -adic completion of  $\mathbb{H}$  with  $H((\frac{1}{t}))$ . Therefore we deduce that:

$$f = \frac{\alpha(f)}{t^k} + \sum_{i \geq k+1} f_i \frac{1}{t^i}, k \leq -1$$

where  $\alpha(f) \in H^\times$ , and  $f_i \in H$ , for all  $i \geq k+1$ .

Let  $\exp_\phi$  be the unique element in  $H\{\{\tau\}\}$  such that  $\exp_\phi \equiv 1 \pmod{\tau}$  and:

$$\forall a \in A, \quad \exp_\phi a = \phi_a \exp_\phi.$$

Write  $\exp_\phi = \sum_{i \geq 0} e_i(\phi) \tau^i$ , then by [16], Corollary 7.4.9, we obtain:

$$H = K(e_i(\phi), i \geq 0).$$

Observe that  $\exp_\phi$  induces an entire function on  $\mathbb{C}_\infty$ , and there exists  $\alpha \in \mathbb{C}_\infty^\times$  and  $I \in \mathcal{I}(A)$  such that:

$$\forall z \in \mathbb{C}_\infty, \quad \exp_\phi(z) = \sum_{i \geq 0} e_i(\phi) z^{q^i} = z \prod_{a \in I \setminus \{0\}} (1 - \frac{z}{\alpha a}).$$

Furthermore, we have (see for example [22], Proposition 0.3.6):

$$\forall i \geq 0, e_i(\phi) = \frac{1}{f \cdots f^{(i-1)} \mid_{\xi^{(i)}}}.$$

Thakur proved that if  $e_n(\phi) = 0$ , then  $n \in \{2, \dots, g-1\}$  ([22], proof of Theorem 3.2), and if  $K$  has a place of degree one then  $\forall n \geq 0, e_n(\phi) \neq 0$ .

Let  $W(B) = \oplus_{i \geq 0} B f \cdots f^{(i-1)}$ . Then  $W(B)$  is a finitely generated  $B \otimes_{\mathbb{F}_q} A = B[\rho(A)]$ -module of rank one (see for example [4], Lemma 4.4). Furthermore,

$$\forall x \in W(B), \quad f x^{(1)} \in W(B).$$

Let  $I \in \mathcal{I}(A)$ . Let  $\phi_I \in H\{\tau\}$  such that the coefficient of its term of highest degree in  $\tau$  is one, and such that:

$$\sum_{a \in I} H\{\tau\} \phi_a = H\{\tau\} \phi_I.$$

Then, we get:

$$\begin{aligned} \deg_\tau \phi_I &= \deg I, \\ \text{Ker } \phi_I \mid_{\mathbb{C}_\infty} &= \cap_{a \in I} \text{Ker } \phi_a \mid_{\mathbb{C}_\infty}, \\ \phi_I &\in B\{\tau\}. \end{aligned}$$

We denote by  $\psi_\phi(I) \in B \setminus \{0\}$  the constant term of  $\phi_I$ . We set:

$$u_I = \sum_{j=0}^{\deg I} \phi_{I,j} f \cdots f^{(j-1)} \in W(B),$$

where  $\phi_I = \sum_{j=0}^{\deg I} \phi_{I,j} \tau^j$ .

**Lemma 3.1.** *Let  $I, J$  be two non-zero ideals of  $A$ . We have:*

$$\begin{aligned} u_I \mid_\xi &= \psi_\phi(I), \\ \sigma_I(f)u_I &= fu_I^{(1)}, \\ u_{IJ} &= \sigma_I(u_J)u_I. \end{aligned}$$

*Proof.* In [4], Lemma 4.6, we only gave a sketch of the proof of the above results. We give here a detailed proof for the convenience of the reader.

Observe that:

$$\forall i \geq 1, \quad (f \cdots f^{(i-1)}) = V^{(i)} - V + \sum_{k=0}^{i-1} (\xi^{(k)}) - \sum_{k=0}^{i-1} (\infty^{(k)}).$$

Since  $\xi$  does not belong to the support of  $V$ , we deduce that:

$$u_I \mid_\xi = \psi_\phi(I).$$

Note that we have a natural isomorphism of  $B$ -modules:

$$\gamma : \begin{cases} W(B) & \xrightarrow{\sim} B\{\tau\} \\ \forall i \geq 0, f \cdots f^{(i-1)} & \longmapsto \tau^i. \end{cases}$$

For all  $x \in W(B)$  and for all  $a \in A$ , we have:

$$\begin{aligned} \gamma(fx^{(1)}) &= \tau\gamma(x), \\ \gamma(\rho(a)x) &= \gamma(x)\phi_a. \end{aligned}$$

In particular  $\gamma$  is an isomorphism of  $B[\rho(A)]$ -modules, and since  $W(B)$  is a finitely generated  $B[\rho(A)]$ -module of rank one, this is also the case of  $B\{\tau\}$ . Write  $f = \frac{\sum_i \rho(a_i)b_i}{\sum_k \rho(c_k)d_k}$ , for some  $a_i, c_k \in A$ ,  $b_i, d_k \in B$ , we have the following equality in  $B\{\tau\}$ :

$$\sum_i b_i \phi_{a_i} = \sum_k d_k \tau \phi_{c_k}.$$

For  $\sigma \in G$ , we set:

$$W_\sigma(B) = \oplus_{i \geq 0} B\sigma(f) \cdots \sigma(f)^{(i-1)}.$$

We have again an isomorphism of  $B[\rho(A)]$ -modules:

$$\gamma_\sigma : W_\sigma(B) \simeq B\{\tau\}.$$

Again,

$$\forall x \in W_\sigma(B), \forall a \in A, \quad \gamma_\sigma(\rho(a)x) = \gamma_\sigma(x)\phi_a^\sigma.$$

Let  $I$  be a non-zero ideal of  $A$ , and let  $\sigma = \sigma_I \in G$ . We start from the relation:

$$\sum_i b_i^\sigma \phi_{a_i}^\sigma = \sum_k d_k^\sigma \tau \phi_{c_k}^\sigma.$$

We multiply on the right by  $\phi_I$ , to obtain (see [16], Theorem 7.4.8):

$$\sum_i b_i^\sigma \phi_I \phi_{a_i} = \sum_k d_k^\sigma \tau \phi_I \phi_{c_k}.$$

Since  $\gamma(fu_I^{(1)}) = \tau\phi_I$ , we get:

$$\left(\sum_i \rho(a_i)b_i^\sigma\right) \cdot \gamma(u_I) = \left(\sum_k d_k^\sigma \rho(c_k)\right) \cdot \gamma(fu_I^{(1)}).$$



In other words, we have proved:

$$\sigma(f)u_I = fu_I^{(1)}.$$

Now, let  $J$  be a non-zero ideal of  $A$ . We have:

$$\gamma(u_{IJ}) = \phi_{IJ} = \phi_J^\sigma \phi_I.$$

Since  $\forall i \geq 0, \sigma(f \cdots f^{(i-1)})u_I = f \cdots f^{(i-1)}u_I^{(i)}$ , we get:

$$\gamma(u_J^\sigma u_I) = \phi_J^\sigma \phi_I.$$

It implies:

$$u_{IJ} = \sigma(u_J)u_I.$$

□

**Corollary 3.2.** *We have:*

$$\text{Sht} = \{\sigma(f), \sigma \in G\}.$$

Furthermore, for  $\sigma \in G$ ,  $\phi^\sigma$  is the Drinfeld module associated to the shtuka function  $\sigma(f)$ .

*Proof.* Let  $\sigma \in G$  and let  $g \in \text{Sht}$  be the shtuka function associated to  $\phi^\sigma$ . By the proof of Lemma 3.1, if  $a'_i, c'_k \in A$ ,  $b'_i, d'_k \in B$  are such that  $\sum_i b'_i \phi_{a'_i}^\sigma = \sum_k d'_k \tau \phi_{c'_k}^\sigma$ , then:

$$g = \frac{\sum_i \rho(a'_i) b'_i}{\sum_k \rho(c'_k) d'_k}.$$

Again, by the proof of Lemma 3.1, we get:

$$g = \sigma(f).$$

□

**Lemma 3.3.** *Let  $\iota_\infty : \mathbb{H} \rightarrow H((\frac{1}{t}))$  be a homomorphism of  $\mathbb{K}$ -algebras corresponding to  $\infty$ . Write  $\iota_\infty(f) = \frac{\alpha(f)}{t^k} + \sum_{i \geq k+1} f_i \frac{1}{t^i} \in H((\frac{1}{t}))$ ,  $\alpha(f) \in H^\times$ ,  $f_i \in H$ ,  $i \geq 0$ ,  $k \leq -1$ . Then:*

$$H = K(\mathbb{F}_\infty, \alpha(f), f_i, i \geq k+1).$$

Furthermore:

$$H_A = K(\mathbb{F}_\infty, \frac{f_i}{\alpha(f)}, i \geq k+1).$$

In particular, there exists  $u(f) \in B^\times$  such that:

- $H = H_A(u(f))$ ,
- $\alpha(f) \equiv \iota_\infty(u(f)) \pmod{H_A^\times}$ ,
- $\mathbb{K}(\frac{f}{u(f)}) = \text{Frac}(H_A \otimes_{\mathbb{F}_q} A)$ .

*Proof.* By Corollary 3.2, since  $|G| = |\text{Sht}|$ , we have:

$$\mathbb{H} = \mathbb{K}(f).$$

Recall that  $H((\frac{1}{t}))$  is isomorphic to the completion of  $\mathbb{H}$  at  $\infty$ . Since  $\infty$  splits totally in  $K(\mathbb{F}_\infty)$  in  $d_\infty$  places, we deduce that the natural map  $\iota_\infty : \mathbb{H} \hookrightarrow H((\frac{1}{t}))$  is  $\text{Gal}(H/K(\mathbb{F}_\infty))$ -equivariant. Thus:

$$H = K(\mathbb{F}_\infty, \alpha(f), f_i, i \geq k+1).$$

If  $I = aA$ ,  $a \in A \setminus \{0\}$ , then  $u_I = \rho(a)$ , so that we have by Lemma 3.1 :

$$\sigma_I(f) = \text{sgn}(a)^{q^{n(\phi)} - q^{n(\phi)+1}} f.$$

In particular:

$$\text{sgn}_{\infty(-1)}(\iota_{\infty(-1)}(f)) \notin \mathbb{F}_\infty^\times.$$

We have  $\alpha(f)^{\frac{q^{d_\infty}-1}{q-1}} \in H_A$ , and  $\frac{f}{\alpha'(f)} \in \text{Frac}(H_A \otimes_{\mathbb{F}_q} A)$ , where  $\alpha'(f) \in H^\times$  is such that  $\iota_\infty(\alpha'(f)) = \alpha(f)$  (observe that  $\iota_\infty|_H \in G$ ). Since  $\mathbb{H} = \mathbb{K}(f)$ , we get the second assertion.

Since  $H/H_A$  is totally ramified at each place of  $H_A$  above  $\infty$ ,  $\frac{B^\times}{(B')^\times}$  is a finite abelian group, where we recall that  $B'$  is the integral closure of  $A$  in  $H_A$ . Now recall that  $H/H_A$  is a cyclic extension of degree  $\frac{q^{d_\infty}-1}{q-1}$ , and  $\mathbb{F}_\infty \subset H_A$ . Let  $\langle \sigma \rangle = \text{Gal}(H_A((B)^\times)/H_A)$ . Then we have an injective homomorphism:

$$\frac{B^\times}{(B')^\times} \hookrightarrow \mathbb{F}_\infty^\times, x \mapsto \frac{x}{\sigma(x)}.$$

The image of this homomorphism is a cyclic group of order dividing  $\frac{q^{d_\infty}-1}{q-1}$ . By the proof [16], Theorem 7.6.4, there exists  $\zeta \in \mathbb{C}_\infty^\times, \zeta^{q-1} \in H$ , such that:

$$\forall a \in A \setminus \{0\}, \zeta \phi_a \zeta^{-1} \in B' \setminus \{0\} \text{ and its highest coefficient is in } (B')^\times.$$

Thus  $\zeta^{q-1} \in B^\times$  and  $H = H_A(\zeta^{q-1})$ . In particular, there exists a group isomorphism:

$$\frac{B^\times}{(B')^\times} \simeq \frac{\mathbb{F}_\infty^\times}{\mathbb{F}_q^\times}.$$

This implies by Kummer Theory that:

$$\alpha(f) \equiv u'(f) \pmod{H_A^\times},$$

for some  $u'(f) \in B^\times$  that generates the cyclic group  $\frac{B^\times}{(B')^\times}$ . Now define  $u(f)$  to be the element in  $B^\times$  such that  $\iota_\infty(u(f)) = u'(f)$ .  $\square$

### 3.2. Special functions.

We fix  ${}^{q^{d_\infty}}\sqrt{-\pi} \in \mathbb{C}_\infty$  a root of the polynomial  $X^{q^{d_\infty}-1} + \pi = 0$ . We consider the period lattice of  $\phi$ :

$$\Lambda(\phi) = \{x \in \mathbb{C}_\infty, \exp_\phi(x) = 0\}.$$

Then  $\Lambda(\phi)$  is a finitely generated  $A$ -module of rank one and we have an exact sequence of  $A$ -modules induced by  $\exp_\phi$ :

$$0 \rightarrow \Lambda(\phi) \rightarrow \mathbb{C}_\infty \rightarrow \phi(\mathbb{C}_\infty) \rightarrow 0,$$

where  $\phi(\mathbb{C}_\infty)$  is the  $\mathbb{F}_q$ -vector space  $\mathbb{C}_\infty$  viewed as an  $A$ -module via  $\phi$ .

**Lemma 3.4.** *We have:*

$$\Lambda(\phi) \subset {}^{q^{d_\infty}}\sqrt{-\pi}^{-q^{n(\phi)}} K_\infty,$$

and for all  $I \in \mathcal{I}(A)$ :

$$\Lambda(\phi^{\sigma_I}) = \psi_\phi(I) I^{-1} \Lambda(\phi).$$

*Proof.* Observe that  $\Lambda(\phi)K_\infty$  is a  $K_\infty$ -vector space of dimension one. Let  $J$  be a non-zero ideal of  $A$ , and let  $\lambda_J \neq 0$  be a generator of the  $A$ -module of  $J$ -torsion points of  $\phi$ . By the proof of [16], Proposition 7.5.16, we have:

$$\lambda_J \in \Lambda(\phi)K_\infty.$$

By class field theory (see [16], section 7.5), we have:

$$E := H(\lambda_J) \subset K_\infty({}^{q^{d_\infty}-1}\sqrt{-\pi}).$$

Furthermore, by [16], Remark 7.5.17,

$$\lambda_J^{q^{d_\infty}-1} \in K_\infty^\times.$$

By local class field theory, for  $x \in K_\infty^\times$ , we have:

$$(x, K_\infty({}^{q^{d_\infty}-1}\sqrt{-\pi})/K_\infty)({}^{q^{d_\infty}-1}\sqrt{-\pi}) = \frac{{}^{q^{d_\infty}-1}\sqrt{-\pi}}{\text{sgn}(x)}.$$

By [16], Corollary 7.5.7, for all  $a \in K^\times, a \equiv 1 \pmod{J}$ , we get:

$$(aA, E/K)(\lambda_J) = \text{sgn}(a)^{-q^{n(\phi)}} \lambda_J.$$

Thus, for all  $a \in K^\times, a \equiv 1 \pmod{J}$ :

$$(a, K_\infty({}^{q^{d_\infty}-1}\sqrt{-\pi})/K_\infty)(\lambda_J) = \text{sgn}(a)^{q^{n(\phi)}} \lambda_J.$$

Therefore, by the approximation Theorem, we get:

$$\forall x \in K_\infty^\times, \quad (x, K_\infty({}^{q^{d_\infty}-1}\sqrt{-\pi})/K_\infty)(\lambda_J) = \text{sgn}(x)^{q^{n(\phi)}} \lambda_J.$$

It implies:

$$\lambda_J \in {}^{q^{d_\infty}-1}\sqrt{-\pi}^{-q^{n(\phi)}} K_\infty.$$

Hence,

$$\Lambda(\phi) \subset {}^{q^{d_\infty}-1}\sqrt{-\pi}^{-q^{n(\phi)}} K_\infty.$$

The second assertion comes from the fact that we have the following equality in  $H\{\{\tau\}\}$ :

$$\phi_I \exp_\phi = \exp_{\phi \sigma_I} \psi_\phi(I).$$

□

Set:

$$L = \rho(K)(\mathbb{F}_\infty)({}^{q^{d_\infty}-1}\sqrt{-\pi}).$$

Then, by the above Lemma,  $H \subset \mathbb{F}_\infty({}^{q^{d_\infty}-1}\sqrt{-\pi}) \subset L$ . Let  $v_\infty : L \rightarrow \mathbb{Q} \cup \{+\infty\}$  be the valuation on  $L$  which is trivial on  $\rho(K)(\mathbb{F}_\infty)$  and such that  $v_\infty({}^{q^{d_\infty}-1}\sqrt{-\pi}) = \frac{1}{q^{d_\infty}-1}$ . Let  $\tau : L \rightarrow L$  be the continuous homomorphism of  $\rho(K)$ -algebras such that:

$$\forall x \in \mathbb{F}_\infty({}^{q^{d_\infty}-1}\sqrt{-\pi}), \quad \tau(x) = x^q.$$

Observe that:

$$\forall x \in L, \quad v_\infty(\tau(x)) = qv_\infty(x).$$

**Lemma 3.5.** *We have:*

$$\text{Ker exp}_\phi|_L = \Lambda(\phi)\rho(K),$$

where  $\Lambda(\phi)\rho(K)$  is the  $\rho(K)$ -vector space generated by  $\Lambda(\phi)$ .

*Proof.* The proof is standard in non-archimedean functional analysis, we give a sketch of the proof for the convenience of the reader. We have:

$$\Lambda(\phi)\rho(K) \subset \text{Ker exp}_\phi |_L.$$

Let:

$$\mathfrak{M} = {}^{q^{d_\infty}}\sqrt{-\pi}\rho(K)(\mathbb{F}_\infty)[[{}^{q^{d_\infty}}\sqrt{-\pi}]].$$

Let  $\log_\phi \in H\{\{\tau\}\}$  such that  $\log_\phi \exp_\phi = \exp_\phi \log_\phi = 1$ . If we write:  $\log_\phi = \sum_{i \geq 0} l_i(\phi)\tau^i$ , then there exists  $C \in \mathbb{R}$  such that, for all  $i \geq 0$ ,  $v_\infty(l_i(\phi)) \geq Cq^i$ . It implies that there exists an integer  $N \geq 0$  such that  $\exp_\phi$  is an isometry on  $\mathfrak{M}^N$ .

Now, select  $\theta \in A \setminus \mathbb{F}_q$ . Then:

$$\text{Ker exp}_\phi |_{\mathbb{F}_\infty[\rho(\theta)]({}^{q^{d_\infty}}\sqrt{-\pi})} = \Lambda(\phi)\mathbb{F}_q[\rho(\theta)].$$

Since  $\rho(A)$  is finitely generated and free as an  $\mathbb{F}_q[\rho(\theta)]$ -module, it implies:

$$\text{Ker exp}_\phi |_{\rho(A)[\mathbb{F}_\infty]({}^{q^{d_\infty}}\sqrt{-\pi})} = \Lambda(\phi)\rho(A).$$

Let  $V$  be the  $\rho(K)$ -vector space generated by  $\rho(A)[\mathbb{F}_\infty]({}^{q^{d_\infty}}\sqrt{-\pi})$ . Then:

$$\text{Ker exp}_\phi |_V = \Lambda(\phi)\rho(K).$$

Let  $x \in \text{Ker exp}_\phi |_L$ , then there exists  $y \in V$  such that:

$$x - y \in \mathfrak{M}^N.$$

Thus,

$$\exp_\phi(y - x) = \exp_\phi(y) \in \mathfrak{M}^N \cap V = \exp_\phi(\mathfrak{M}^N \cap V).$$

Therefore,  $y = z + v$ , for some  $z \in \mathfrak{M}^N \cap V$ , and some  $v \in \Lambda(\phi)\rho(K)$ . It implies that  $x - v \in \mathfrak{M}^N$ , and hence:

$$x = v \in \Lambda(\phi)\rho(K).$$

□

**Lemma 3.6.** *We consider the following  $\rho(K)$ -vector space:*

$$V = \bigcap_{a \in A \setminus \mathbb{F}_q} \exp_\phi\left(\frac{1}{a - \rho(a)}\Lambda(\phi)\rho(K)\right)$$

*Then, we have:*

$$\dim_{\rho(K)} V = 1.$$

*Proof.* For any  $a \in A$ , we set:

$$V_a = \{x \in L, \phi_a(x) = \rho(a)x\}.$$

Then, if  $a \notin \mathbb{F}_q$ , by Lemma 3.5, we have:

$$V_a = \exp_\phi\left(\frac{1}{a - \rho(a)}\Lambda(\phi)\rho(K)\right),$$

and:

$$\dim_{\rho(K)} V_a = \deg a = [K : \mathbb{F}_q(a)].$$

Select  $\theta \in A \setminus \mathbb{F}_q$  such that  $K/\mathbb{F}_q(\theta)$  is a finite separable extension. Let  $b \in A \setminus \mathbb{F}_q$  and let  $P_b(X) \in \mathbb{F}_q[\theta][X]$  be the minimal polynomial of  $b$  over  $\mathbb{F}_q(\theta)$ . Since  $V_\theta$  is an  $A$ -module via  $\phi$  and  $\phi_b$  induces a  $\rho(K)$ -linear endomorphism of  $V_\theta$ , it follows that:

$$\rho(P_b)(\phi_b) = 0.$$

This implies that the minimal polynomial of  $\phi_b$  viewed as an  $\mathbb{F}_q(\rho(\theta))$ -linear endomorphism of  $V_\theta$  is  $\rho(P_b(X))$ . Observe that  $V_\theta$  is the  $\rho(K)$ -vector space generated by:

$$\exp_\phi\left(\frac{1}{\theta - \rho(\theta)}\Lambda(\phi)\mathbb{F}_q(\rho(\theta))\right),$$

and:

$$\dim_{\mathbb{F}_q(\rho(\theta))} \exp_\phi\left(\frac{1}{\theta - \rho(\theta)}\Lambda(\phi)\mathbb{F}_q(\rho(\theta))\right) = \deg \theta.$$

Therefore,  $\rho(P_b(X))$  is the minimal polynomial of  $\phi_b$  viewed as a  $\rho(K)$ -linear endomorphism of  $V_\theta$ .

Select  $\theta' \in A \setminus \mathbb{F}_q$  such that  $K = \mathbb{F}_q(\theta, \theta')$ . Then the characteristic polynomial of  $\phi_{\theta'}$  on the  $\rho(K)$ -vector space  $V_\theta$  is  $\rho(P_{\theta'}(X))$ . Since  $P_{\theta'}(X)$  has simple roots, if  $V' = V_\theta \cap V_{\theta'}$ , we get:

$$\dim_{\rho(K)} V' = 1.$$

Now, let  $b \in A$ , there exists  $x, y \in A[\theta, \theta']$ , such that  $b = \frac{x}{y}$ . Let  $\lambda_b \in \rho(K)$  such  $\phi_b \mid V'$  is the multiplication by  $\lambda_b$ , then for any  $v \in V' \setminus 0$ , we have:

$$\rho(y)\lambda_b v = \phi_y b v = \rho(x)v$$

It follows that:

$$\lambda_b = \rho(b).$$

□

Let  $\text{sgn} : \rho(K)(\mathbb{F}_\infty)((\pi))^\times \rightarrow \rho(K)(\mathbb{F}_\infty)^\times$  be the group homomorphism such that  $\text{Ker sgn} = \pi^\mathbb{Z} \times (1 + \pi\rho(K)(\mathbb{F}_\infty)[[\pi]])$ , and  $\text{sgn} \mid_{\rho(K)(\mathbb{F}_\infty)^\times} = \text{Id} \mid_{\rho(K)(\mathbb{F}_\infty)^\times}$ . Let  $\pi_* = (\sqrt[q^{d_\infty-1}]{-\pi})^{(q-1)q^{n(\phi)}}$ .

**Lemma 3.7.** *We have:*

$$\begin{aligned} f\pi_* &\in \rho(K)(\mathbb{F}_\infty)((\pi)), \\ v_\infty(f) &\equiv -\frac{(q-1)q^{n(\phi)}}{q^{d_\infty}-1} \pmod{(q-1)\mathbb{Z}}, \end{aligned}$$

and:

$$N_{\rho(K)(\mathbb{F}_\infty)/\rho(K)}(\text{sgn}(f\pi_*)) = 1.$$

*Proof.*

1) Recall that:

$$V = \bigcap_{a \in A \setminus \mathbb{F}_q} \exp_\phi\left(\frac{1}{a - \rho(a)}\Lambda(\phi)\rho(K)\right).$$

By Lemma 3.4, we have:

$$V \subset (\sqrt[q^{d_\infty-1}]{-\pi})^{-q^{n(\phi)}} \rho(K)(\mathbb{F}_\infty)((\pi)).$$

Thus, by Lemma 3.6, there exists  $U \in (\sqrt[q^{d_\infty-1}]{-\pi})^{-q^{n(\phi)}} \rho(K)(\mathbb{F}_\infty)((\pi)) \setminus \{0\}$ , such that:

$$\forall a \in A, \quad \phi_a(U) = \rho(a)U.$$

Write  $f = \frac{\sum_i \rho(a_i)b_i}{\sum_k \rho(a'_k)b'_k}$ ,  $a_i, a'_k \in A$ ,  $b_i, b'_k \in B$ . Then, by the proof of Lemma 3.1, we have:

$$\sum_i b_i \phi_{a_i} = \sum_k b'_k \tau \phi_{a'_k}.$$

Thus,

$$\left(\sum_i \rho(a_i)b_i\right)U = \left(\sum_k \rho(a'_k)b'_k\right)\tau(U).$$

Therefore:

$$\tau(U) = fU.$$

In particular,

$$\{x \in L, \tau(x) = fx\} = \rho(K)U.$$

We also get:

$$f \in \pi_*^{-1} \rho(K)(\mathbb{F}_\infty)((\pi)).$$

2) Let  $F = f\pi_* \in \rho(K)(\mathbb{F}_\infty)((\pi))$ . Set  $R = U \left( {}^{q^{d_\infty}}\sqrt{-\pi} \right)^{q^{n(\phi)}} \in \rho(K)(\mathbb{F}_\infty)((\pi))$ . We have:

$$\tau(R) = FR.$$

Let  $i_0 = v_\infty(F) \in \mathbb{Z}$ , and write:

$$F = \sum_{i \geq i_0} F_i(-\pi)^i, F_i \in \rho(K)(\mathbb{F}_\infty).$$

Let  $\lambda = F_{i_0}$ . Set:

$$\alpha = {}^{q^{-1}}\sqrt{-\pi}^{i_0} \left( \prod_{i \geq 0} \frac{F^{(i)}}{\lambda^{(i)}(-\pi)^{q^i i_0}} \right)^{-1} \in L^\times,$$

where  ${}^{q^{-1}}\sqrt{-\pi} = \left( {}^{q^{d_\infty}}\sqrt{-\pi} \right)^{\frac{q^{d_\infty}-1}{q-1}}$ . Then clearly:

$$\tau(\alpha) = \frac{F}{\lambda} \alpha.$$

Thus:

$$\tau\left(\frac{R}{\alpha}\right) = \lambda \frac{R}{\alpha}.$$

This implies:

$$R = \mu \alpha, \mu \in \rho(K)(\mathbb{F}_\infty)^\times.$$

In particular,  $i_0 \equiv 0 \pmod{q-1}$ , i.e.  $v_\infty(f) \equiv -\frac{(q-1)q^{n(\phi)}}{q^{d_\infty}-1} \pmod{q-1}$ . Also:

$$\text{sgn}(R) = \mu \text{sgn}(\alpha).$$

Since  $\text{sgn}(\alpha) = (-1)^{\frac{i_0}{q-1}}$ , we get:

$$\frac{\tau(\mu)}{\mu} = \lambda.$$

□

We set:

$$\mathbb{T} := \rho(A)[\mathbb{F}_\infty]({}^{q^{d_\infty}}\sqrt{-\pi}) \subset L.$$

Then  $\mathbb{T}$  is complete with respect to the valuation  $v_\infty$ , and:

$$\{x \in \mathbb{T}, \tau(x) = x\} = \rho(A).$$

Furthermore, we have (see the proof of Lemma 3.5):

$$\text{Ker exp}_\phi|_{\mathbb{T}} = \Lambda(\phi)\rho(A).$$

Let  $\text{ev} : \rho(A)[\mathbb{F}_\infty] \rightarrow \overline{\mathbb{F}}_q \subset \mathbb{C}_\infty$  be a homomorphism of  $\mathbb{F}_\infty$ -algebras. Such a homomorphism induces a continuous homomorphism  $\mathbb{F}_\infty((\sqrt[q^{d_\infty}]{-\pi}))$ -algebras:

$$\text{ev} : \mathbb{T} \rightarrow \mathbb{C}_\infty.$$

We denote by  $\mathcal{E}$  the set of such continuous homomorphisms from  $\mathbb{T}$  to  $\mathbb{C}_\infty$ .

**Proposition 3.8.** *We have:*

$$f \in \mathbb{T}^\times, \\ \text{sgn}(f\pi_*) \in \rho(A)[\mathbb{F}_\infty]^\times.$$

Furthermore there exists  $U \in \mathbb{T} \setminus \{0\}$  such that:

$$\{x \in L, \tau(x) = fx\} = U\rho(K).$$

If  $d_\infty = 1$ , then  $\text{sgn}(f\pi_*) = 1$ , and we can take:

$$U = \sqrt[q^{d_\infty}]{-\pi}^{-1} \sqrt[q^{d_\infty}]{-\pi}^{i_0} \left( \prod_{i \geq 0} \frac{(f\pi_*)^{(i)}}{(-\pi)^{q^i i_0}} \right)^{-1} \in \mathbb{T}^\times,$$

where  $i_0 := v_\infty(f\pi_*)$ .

*Proof.* Recall that  $f \in \mathbb{H} \subset L$ . Let  $P$  be a point in  $\bar{X}(\overline{\mathbb{F}}_q)$  above a maximal ideal of  $\rho(A)$ . Then  $P$  is above a maximal ideal of  $\rho(A)[\mathbb{F}_\infty]$  which can be viewed as the kernel of some homomorphism of  $\mathbb{F}_\infty$ -algebras  $\text{ev} : \rho(A)[\mathbb{F}_\infty] \rightarrow \overline{\mathbb{F}}_q$ . Since the field of constants of  $H$  is  $\mathbb{F}_\infty$ , we deduce that  $\text{ev}$  can be uniquely extended to a homomorphism of  $H$ -algebras:

$$\text{ev} : \rho(A)[H] \rightarrow \mathbb{C}_\infty.$$

Furthermore, the kernel of the above homomorphism corresponds to  $P \cap \mathbb{H}$  (recall that  $\mathbb{H} = \text{Frac}(\rho(A)[H])$ ). Then  $\text{ev}$  extends to a continuous homomorphism of  $\mathbb{F}_\infty((\sqrt[q^{d_\infty}]{-\pi}))$ -algebras:

$$\text{ev} : \mathbb{T} \rightarrow \mathbb{C}_\infty.$$

We deduce that, by [22], Lemma 1.1, for any  $\text{ev} \in \mathcal{E}$ ,  $\text{ev}(f)$  is well-defined. Thus  $f \in \mathbb{T}$ . Therefore, by Lemma 3.7, we have:

$$f \in \pi_*^\mathbb{Z} \times (\text{sgn}(f\pi_*) + \pi\rho(A)[\mathbb{F}_\infty][[\pi]]),$$

where  $\text{sgn}(f\pi_*) \in \rho(A)[\mathbb{F}_\infty]$  is such that:

$$N_{\rho(K)(\mathbb{F}_\infty)/\rho(K)}(\text{sgn}(f\pi_*)) = 1.$$

Thus:

$$\text{sgn}(f\pi_*) \in \rho(A)[\mathbb{F}_\infty]^\times,$$

and there exists  $\mu \in \rho(A)[\mathbb{F}_\infty] \setminus \{0\}$  such that:

$$\text{sgn}(f\pi_*) = \frac{\tau(\mu)}{\mu}.$$

In particular,  $f \in \mathbb{T}^\times$ . Furthermore, there exists a non-zero ideal  $I$  of  $A$  such that:

$$\mu\rho(A)[\mathbb{F}_\infty] = \rho(I)\rho(A)[\mathbb{F}_\infty].$$

Now, we use the proof of Lemma 3.7. We put  $i_0 = v_\infty(f\pi_*)$  (observe that  $i_0 \equiv 0 \pmod{q-1}$ ) and set:

$$U = \mu\alpha \sqrt[q^{d_\infty}]{-\pi}^{-q^{n(\phi)}},$$

where :

$$\alpha = {}^{q^{-1}\sqrt{-\pi}}_{i_0} \left( \prod_{i \geq 0} \frac{(f\pi_*)^{(i)}}{\text{sgn}(f\pi_*)^{(i)} (-\pi)^{q^i i_0}} \right)^{-1} \in \mathbb{T}^\times.$$

Then:

$$\tau(U) = fU,$$

$$U \in \mathbb{T}.$$

Note that  $U$  is well-defined modulo  $\rho(K)^\times$  and if  $d_\infty = 1$ , then  $U \in \mathbb{T}^\times$ .  $\square$

**Definition 3.9.** A non-zero element in  $\{x \in L, \tau(x) = fx\}$  will be called a *special function* attached to the shtuka function  $f$ .

**Remark 3.10.** Let  $M = \{x \in \mathbb{T}, \tau(x) = fx\}$ . Then, by the above Proposition, there exists  $U \in \mathbb{T} \setminus \{0\}$  such that:

$$U\rho(A) \subset M \subset U\rho(K).$$

Furthermore (see the proof of Lemma 3.7):

$$M = \bigcap_{a \in A \setminus \mathbb{F}_q} \exp_\phi \left( \frac{1}{a - \rho(a)} \Lambda(\phi) \rho(A) \right).$$

Thus  $M$  is a finitely generated  $\rho(A)$ -module of rank one. When  $d_\infty = 1$ , the above Proposition tells us that  $M$  is a free  $\rho(A)$ -module. In general, we have:

$$M = U' \rho(\mathcal{B}),$$

where  $\mathcal{B} \in \mathcal{I}(A)$ ,  $U' \in L^\times$ , and  $M = U'' \rho(\mathcal{B}')$  if and only if  $U' = xU''$  where  $x \in \rho(K)^\times$  is such that  $x\mathcal{B} = \mathcal{B}'$ .

Let  $I$  be a non-zero ideal of  $A$ , and let  $\sigma = \sigma_I \in G$ . Recall that, by Lemma 3.1, we have:

$$\sigma(f) = f \frac{\tau(u_I)}{u_I}.$$

Now observe that  $u_I \in \mathbb{T}$ ,  $\frac{\tau(u_I)}{u_I} \in \mathbb{T}^\times$ , but in general we don't have  $u_I \in \mathbb{T}^\times$ . By Lemma 3.1, we have:

$$\frac{u_I}{\rho(x_I)} \in \mathbb{T}^\times,$$

where  $I^n = x_I A$ ,  $n$  being the order of  $I$  in  $\text{Pic}(A)$ . Thus:

$$M_\sigma := \{x \in \mathbb{T}, \tau(x) = \sigma(f)x\} = \frac{\rho(x_I)}{u_I} M.$$

We leave open the following question: is  $M$  a free  $\rho(A)$ -module ? We will show in section 4 that the answer is positive if  $g = 0$ .

### 3.3. The period $\tilde{\pi}$ .

By Lemma 2.2, and Lemma 3.4, let  $f$  be the unique shtuka function in  $\text{Sht}$  such that, if  $\phi$  is the Drinfeld module associated to  $f$ , we have:

$$\text{Ker exp}_\phi |_L = \tilde{\pi} A[\rho(A)],$$

where  $\tilde{\pi} \in {}^{q^{d_\infty} - 1}_{\sqrt{-\pi}} - q^{n(\phi)} K_\infty$ ,  $\text{sgn}(\tilde{\pi} ({}^{q^{d_\infty} - 1}_{\sqrt{-\pi}})^{q^{n(\phi)}}) = 1$ .



**Proposition 3.11.** *There exists  $\theta \in A \setminus \mathbb{F}_q$ ,  $a \in A[\rho(A)]$ , and a special function  $U \in \mathbb{T}$ , such that for all  $i \geq 0$  :*

$$\frac{\rho(\theta) - \theta^{q^i}}{a^{(i)}} U \mid_{\xi^{(i)} = e_i(\phi) \tilde{\pi}^{q^i}}.$$

*In particular, for any special function  $U'$  associated to  $f$ , we have :*

$$\forall i \geq 0, \quad f^{(i)} U' \mid_{\xi^{(i)} \in \tilde{\pi}^{q^i} H}.$$

*Proof.* Let  $\mathbb{A} = A[\rho(K)]$ . We still denote by  $\rho$  the obvious  $\rho(K)$ -linear map  $\mathbb{A} \rightarrow \rho(K)$ . We observe that:

$$\text{Ker} \rho = \sum_{a \in A} (a - \rho(a)) \mathbb{A}.$$

We also observe that there exists  $\theta \in A \setminus \mathbb{F}_q$  such that  $\rho(\theta) - \theta \in \text{Ker} \rho \setminus (\text{Ker} \rho)^2$ . Set  $z = \rho(\theta)$ . Then  $z - \theta$  has a zero of order one at  $\xi$  (observe that  $z - \theta^{q^i}$  has a zero of order one at  $\xi^{(i)}$ ). Note that  $K/\mathbb{F}_q(\theta)$  is a finite separable extension, therefore there exists  $y \in A$  such that  $K = \mathbb{F}_q(\theta, y)$ . Let  $P(X) \in \mathbb{F}_q[\theta][X]$  be the minimal polynomial of  $y$  over  $\mathbb{F}_q(\theta)$  and set:

$$a = \frac{P(X)}{X - y} \mid_{X=\rho(y)} \in A[\rho(A)] \subset \mathbb{A}.$$

Since  $P(X)$  has a zero of order one at  $y$ , we have:

$$a \notin \text{Ker} \rho.$$

Let's set:

$$U = \exp_{\phi} \left( \frac{a}{z - \theta} \tilde{\pi} \right) \in \mathbb{T}.$$

Since  $\frac{a}{z - \theta} \notin \mathbb{A}$ , we have:

$$U \neq 0.$$

Furthermore, observe that  $\mathbb{F}_q[\theta, y] \subset A \subset \text{Frac}(\mathbb{F}_q[\theta, y])$ . Thus:

$$\forall b \in A, \quad \phi_b(U) = \rho(b)U.$$

We conclude that:

$$U \in (\{x \in L, \tau(x) = fx\} \setminus \{0\}) \cap \mathbb{T}.$$

Let's set:

$$\delta = \frac{a}{z - \theta}.$$

We have:

$$U = \sum_{i \geq 0} \delta^{(i)} e_i(\phi) \tilde{\pi}^{q^i}.$$

We therefore get:

$$\forall i \geq 0, \quad (\delta^{-1})^{(i)} U \mid_{\xi^{(i)} = e_i(\phi) \tilde{\pi}^{q^i}}.$$

The last assertion comes from the fact that  $f^{(i)}$  has a zero of order at least one at  $\xi^{(i)}$ . □

We refer the reader to [2] for the explicit construction of  $f$  in the case  $d_{\infty} = 1$ , and to [17] for the explicit construction of the special functions attached to  $f$  in the case  $g = 1$  and  $d_{\infty} = 1$ .

4. A BASIC EXAMPLE: THE CASE  $g = 0$ 

In this section, we assume that the genus of  $K$  is zero. Let's select  $x \in K$  such that  $K = \mathbb{F}_q(x)$  and  $v_\infty(x) = 0$ . Let  $P_\infty(x) \in \mathbb{F}_q[x]$  be the monic irreducible polynomial corresponding to  $\infty$ , then  $\deg_x P_\infty(x) = d_\infty$ . Let  $\text{sgn} : K_\infty^\times \rightarrow \mathbb{F}_\infty^\times$  be the sign function such that  $\text{sgn}(P_\infty(x)) = 1$ . Then:

$$A = \left\{ \frac{f(x)}{P_\infty(x)^k}, k \in \mathbb{N}, f(x) \in \mathbb{F}_q[x], f(x) \not\equiv 0 \pmod{P_\infty(x)}, \deg_x(f(x)) \leq kd_\infty \right\}.$$

Observe that:

$$\text{Pic}(A) \simeq \frac{\mathbb{Z}}{d_\infty \mathbb{Z}}.$$

Let  $P$  be the maximal ideal of  $A$  which corresponds to the pole of  $x$ , i.e.  $P = \left\{ \frac{f(x)}{P_\infty(x)^k}, k \in \mathbb{N}, f(x) \in \mathbb{F}_q[x], f(x) \not\equiv 0 \pmod{P_\infty(x)}, \deg_x(f(x)) < kd_\infty \right\}$ , the order of  $P$  in  $\text{Pic}(A)$  is exactly  $d_\infty$ , and  $P^{d_\infty} = \frac{1}{P_\infty(x)}A$ . We also observe that the Hilbert class field of  $A$  is  $K(\mathbb{F}_\infty)$ . Let  $\zeta = \text{sgn}(x) \in \mathbb{F}_\infty^\times$ . Then  $P_\infty(\zeta) = 0$ . Note that:

$$\begin{aligned} v_\infty(x - \zeta) &= 1, \\ \text{sgn}(x - \zeta) &= P'_\infty(\zeta)^{-1}. \end{aligned}$$

The integral closure of  $A$  in  $K(\mathbb{F}_\infty)$  is  $A[\mathbb{F}_\infty]$ . The abelian group  $A[\mathbb{F}_\infty]^\times$  is equal to:

$$\mathbb{F}_\infty^\times \prod_{k=1}^{d_\infty-1} \left( \frac{x - \zeta}{x - \zeta^{q^k}} \right)^\mathbb{Z}.$$

We know that  $A[\mathbb{F}_\infty]$  is a principal ideal domain and we have:

$$PA[\mathbb{F}_\infty] = \frac{1}{x - \zeta} A[\mathbb{F}_\infty].$$

Furthermore  $B = A[\mathbb{F}_\infty][u]$ , where  $u \in B^\times$  is such that:

$$u^{\frac{q^{d_\infty}-1}{q-1}} = \prod_{k=0}^{d_\infty-1} \frac{\zeta - x^{q^k}}{\zeta^{q^k} - x^{q^k}}.$$

Indeed, using Thakur Gauss sums ([23]), there exists  $g \in \overline{K}$  such that  $K(\mathbb{F}_\infty, g)/K$  is a finite abelian extension and:

$$g^{q^{d_\infty}-1} = \prod_{k=0}^{d_\infty-1} (\zeta - x^{q^k}).$$

Furthermore  $K(\mathbb{F}_\infty, g)/K$  is unramified outside  $\infty$  and the pole of  $x$ , and  $P_\infty(x)$  is a local norm for every place of  $K(\mathbb{F}_\infty, g)$  above  $\infty$ .

Let  $z = \rho(x) \in \rho(K)^\times$ . Then:

$$\mathbb{H} = H(z).$$

Let  $Q \in \bar{X}(\mathbb{F}_q)$  be the unique point which is a pole of  $z$ , then:

$$(z - x) = (\xi) - (Q).$$

We choose  $\infty$  to be the point of  $\bar{X}(\mathbb{F}_\infty)$  which is the zero of  $z - \zeta$ . Then:

$$\left( \frac{z - x}{z - \zeta} \right) = (\xi) - (\infty).$$

We easily deduce that if  $f$  is a shtuka function relative to  $\infty$  (note that  $f$  is well-defined modulo  $\{x \in \mathbb{F}_\infty^\times, x^{\frac{q^{d_\infty}-1}{q-1}} = 1\}$ ), then  $f$  is of the form:

$$\frac{z-x}{z-\zeta}v, v \in H^\times.$$

Let  $\theta = \frac{1}{P_\infty(x)} \in A$ . Then:

$$\begin{aligned} \text{sgn}(\theta) &= 1, \\ \deg \theta &= d_\infty. \end{aligned}$$

Let  $\phi$  be the Drinfeld module attached to  $f$ , then:

$$\phi_\theta = \theta + \dots + \tau^{d_\infty}.$$

We have:

$$f \dots f^{(d_\infty-1)} = \frac{\prod_{k=0}^{d_\infty-1} (z - x^{q^k})}{P_\infty(z)} v^{\frac{q^{d_\infty}-1}{q-1}}.$$

We get:

$$1 = \prod_{k=0}^{d_\infty-1} (\zeta - x^{q^k}) v^{\frac{q^{d_\infty}-1}{q-1}}.$$

Thus:

$$(vg^{q-1})^{\frac{q^{d_\infty}-1}{q-1}} = 1,$$

So that,

$$f = \frac{z-x}{z-\zeta} g^{1-q} \zeta',$$

where  $\zeta' \in \mathbb{F}_\infty^\times$  is such that:

$$(\zeta')^{\frac{q^{d_\infty}-1}{q-1}} = 1.$$

Furthermore, if we write  $\exp_\phi = \sum_{i \geq 0} e_i(\phi) \tau^i$ ,  $e_i(\phi) \in H$ , then:

$$e_i(\phi) = g^{q^i-1} (\zeta')^{-\frac{q^i-1}{q-1}} \prod_{k=0}^{i-1} \frac{x^{q^i} - \zeta^{q^k}}{x^{q^i} - x^{q^k}}.$$

We also deduce that:

$$\forall a \in A, \phi_a = a + \dots + \text{sgn}(a) \tau^{\deg a}.$$

Recall that  $H \subset \mathbb{C}_\infty$ , and  $v_\infty(x - \zeta) = 1$ . We now work in

$$L = \mathbb{F}_\infty(z) \left( (\sqrt[q^{d_\infty}-1]{-P_\infty(x)}) \right).$$

Recall that  $g$  is the Thakur-Gauss sum associated to  $\text{sgn}$ , i.e. let  $C : \mathbb{F}_q[x] \rightarrow \mathbb{F}_q[x]\{\tau\}$  be the homomorphism of  $\mathbb{F}_q$ -algebras such that  $C_x = x + \tau$ , we have chosen  $\lambda \in H \setminus \{0\}$  such that  $C_{P_\infty(x)}(\lambda) = 0$ , and:

$$g = - \sum_{y \in \mathbb{F}_q[x] \setminus \{0\}, \deg_x y < d_\infty} \text{sgn}(y)^{-1} C_y(\lambda).$$

Furthermore,  $\lambda$  is chosen is such a way that:

$$\lambda \in \sqrt[q^{d_\infty}-1]{-P_\infty(x)} K_\infty,$$

$$\text{sgn}\left(\frac{\lambda}{\sqrt[q^{d_\infty}-1]{-P_\infty(x)}}\right) = 1.$$

Thus:

$$\operatorname{sgn}\left(\frac{g}{\sqrt[q^{d_\infty}-1]{-P_\infty(x)}}\right) = 1.$$

Recall also that:

$$\mathbb{T} = \rho(A)[\mathbb{F}_\infty]((\sqrt[q^{d_\infty}-1]{-P_\infty(x)})).$$

We can choose  $f$  such that  $\zeta' = 1$ , i.e.  $f = \frac{z-x}{z-\zeta}g^{1-q}$ . Now, recall that:

$$f, \frac{z-x}{z-\zeta} \in \mathbb{T}^\times.$$

Set:

$$U = \prod_{i \geq 0} \left(1 + \frac{(\zeta - x)^{q^i}}{z - \zeta^{q^i}}\right)^{-1} \in L^\times.$$

Then:

$$U \in \mathbb{T}^\times.$$

Furthermore:

$$\tau(U) = \frac{z-x}{z-\zeta}U.$$

Let's set:

$$\omega = g^{-1}U,$$

Then:

$$\begin{aligned} \tau(\omega) &= f\omega, \\ \operatorname{sgn}(\omega \sqrt[q^{d_\infty}-1]{-P_\infty(x)}) &= 1, \\ \omega &\in \mathbb{T}^\times, \\ \{x \in \mathbb{T}, \tau(x) = fx\} &= \omega\rho(A). \end{aligned}$$

Finally observe that:

$$(z-x)\omega \mid_\xi = g^{-1}(x-\zeta) \prod_{i \geq 1} \left(1 + \frac{(\zeta-x)^{q^i}}{x-\zeta^{q^i}}\right)^{-1}.$$

Thus, there exists  $b \in K^\times$ ,  $\operatorname{sgn}(b) = 1$ ,  $\zeta'$  a root of  $P_\infty(x)$ , such that:

$$\tilde{\pi} = bg'^{-1}(x-\zeta') \prod_{i \geq 1} \left(1 + \frac{(\zeta'-x)^{q^i}}{x-(\zeta')^{q^i}}\right)^{-1},$$

for some well-chosen Thakur Gauss sum  $g'$  relative to a twist of  $\operatorname{sgn}$ .

Let's treat the elementary (and well-known, see [3], and especially the proof of Lemma 2.5.4) case  $d_\infty = 1$ . Then  $A = \mathbb{F}_q[\theta]$  for some  $\theta \in K$ ,  $\operatorname{sgn}(\theta) = 1$ . Let's take  $x = \frac{\theta+1}{\theta}$ . Then  $P_\infty(x) = x - 1$ , and  $\zeta = 1$ . In that case:

$$g = \sqrt[q^{-1}]{-P_\infty(x)} = \sqrt[q^{-1}]{-\frac{1}{\theta}}.$$

We get:

$$f = \frac{z-x}{z-1}g^{1-q} = t - \theta,$$

where  $t = \rho(\theta)$ . We have:

$$\phi_{\frac{1}{P_\infty(x)}} = \phi_\theta = \theta + \tau.$$

We get:

$$\omega = {}^{q-1}\sqrt{-\theta} \prod_{i \geq 0} (1 - \frac{t}{\theta^{q^i}})^{-1} \in \mathbb{T} = \mathbb{F}_q[t](({}^{q-1}\sqrt{\frac{-1}{\theta}})).$$

In this case  $\phi$  is standard, thus we have:

$$\text{Ker exp}_\phi = \tilde{\pi}A,$$

for  $\tilde{\pi} \in {}^{q-1}\sqrt{-\theta}K_\infty$ ,  $\text{sgn}(\tilde{\pi} {}^{q-1}\sqrt{\frac{-1}{\theta}}) = 1$ . Let's set:

$$\omega' = \exp_\phi(\frac{\tilde{\pi}}{f}) \in \mathbb{T} \setminus \{0\}.$$

Then, one has:

$$\phi_\theta(\omega') = \exp_\phi(\theta \frac{\tilde{\pi}}{t - \theta}) = t\omega'.$$

Thus:

$$\forall a \in A, \phi_a(\omega') = \rho(a)\omega'.$$

Therefore there exists  $a \in A \setminus \{0\}$  such that:

$$\omega' = \omega\rho(a).$$

But, since  $\forall i \geq 0, v_\infty(e_i(\phi)) = iq^i$ , by examining the Newton polygon of  $\sum_{i \geq 0} e_i(\phi)\tau^i$ , we get:

$$v_\infty(\tilde{\pi}) = \frac{-q}{q-1}.$$

This implies:

$$v_\infty(\omega' - \frac{\tilde{\pi}}{f}) \geq q - \frac{q}{q-1}.$$

Therefore:

$$\text{sgn}(\omega' {}^{q-1}\sqrt{\frac{-1}{\theta}}) = \text{sgn}(\frac{\tilde{\pi}}{f} {}^{q-1}\sqrt{\frac{-1}{\theta}}) = -1.$$

Thus:

$$\omega' = -\omega.$$

We get:

$$\frac{-\tilde{\pi}}{\theta^2} = (z-x)\omega' \mid_{\xi} = -(z-x)\omega \mid_{\xi}.$$

Thus:

$$(z-x)\omega \mid_{\xi} = \frac{\tilde{\pi}}{\theta^2},$$

and therefore:

$$\tilde{\pi} = \theta^2(z-x)\omega \mid_{\xi} = {}^{q-1}\sqrt{-\theta}\theta \prod_{i \geq 1} (1 - \theta^{1-q^i})^{-1}.$$

5. A RATIONALITY RESULT FOR TWISTED  $L$ -SERIES

Let  $s$  be an integer,  $s \geq 1$ . We introduce:

$$\mathcal{A}_s = A \otimes_{\mathbb{F}_q} \cdots \otimes_{\mathbb{F}_q} A = A^{\otimes s},$$

and set:

$$k_s = \text{Frac}(\mathcal{A}_s).$$

For  $i = 1, \dots, s$ , let  $\rho_i : K \rightarrow k_s$  be the homomorphism of  $\mathbb{F}_q$ -algebras such that  $\forall a \in A, \rho_i(a) = 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \cdots \otimes 1$ , where  $a$  appears at the  $i$ th position. We set:

$$\mathbb{A}_s = A \otimes_{\mathbb{F}_q} k_s,$$

$$\mathbb{K}_s = \text{Frac}(\mathbb{A}_s),$$

$$\mathbb{H}_s = \text{Frac}(B \otimes_{\mathbb{F}_q} k_s).$$

We identify  $H$  with its image  $H \otimes 1$  in  $\mathbb{H}_s$ , and  $k_s$  with its image  $1 \otimes k_s$ . Thus:

$$\mathbb{A}_s = A[k_s].$$

We also identify  $G$  with the Galois group of  $\mathbb{H}_s/\mathbb{K}_s$ . For  $i = 1, \dots, s$ ,  $\rho_i$  induces a homomorphism of  $H$ -algebras:

$$\rho_i : \mathbb{H} \rightarrow \mathbb{H}_s.$$

Let  $\mathbb{K}_{s,\infty}$  be the  $\infty$ -adic completion of  $\mathbb{K}_s$ , i.e.:

$$\mathbb{K}_{s,\infty} = k_s[\mathbb{F}_\infty]((\pi)).$$

We set:

$$\mathbb{H}_{s,\infty} = \mathbb{H}_s \otimes_{\mathbb{K}_s} \mathbb{K}_{s,\infty}.$$

Then we have an isomorphism of  $\mathbb{K}_{s,\infty}$ -algebras:

$$\kappa : \mathbb{H}_{s,\infty} \simeq k_s[\mathbb{F}_\infty]((\pi_*))^{|\text{Pic}(A)|},$$

where we set  $\pi_* := \frac{q^{d_\infty-1}}{q-1} \sqrt{-\pi}$ .

Let  $V$  be a finite dimensional  $\mathbb{K}_{s,\infty}$ -vector space. An  $\mathbb{A}_s$ -module  $M$ ,  $M \subset V$ , will be called an  $\mathbb{A}_s$ -lattice in  $V$ , if  $M$  is a finitely generated  $\mathbb{A}_s$ -module which is discrete in  $V$  and such that  $M$  contains a  $\mathbb{K}_{s,\infty}$ -basis of  $V$ . For example,  $\mathbb{B}_s := B[k_s]$  is an  $\mathbb{A}_s$ -lattice in  $\mathbb{H}_{s,\infty}$ .

Let  $\phi \in \text{Drin}$  and let  $f$  be its associated shtuka function. For  $i = 1, \dots, s$  we set:

$$f_i = \rho_i(f).$$

Let  $\tau : \mathbb{H}_{s,\infty} \rightarrow \mathbb{H}_{s,\infty}$  be the continuous homomorphism of  $k_s$ -algebras such that:

$$\forall x \in H \otimes_K K_\infty, \quad \tau(x) = x^q.$$

Let  $\varphi_s : \mathbb{A}_s \rightarrow \mathbb{H}_s\{\tau\}$  be the homomorphism of  $k_s$ -algebras such that:

$$\forall a \in A, \quad \varphi_{s,a} = \sum_{k=0}^{\deg a} \phi_{a,k} \prod_{i=1}^s \prod_{j=0}^{k-1} f_i^{(j)} \tau^k.$$

We consider:

$$\exp_{\varphi_s} = \sum_{k \geq 0} e_k(\phi) \prod_{i=1}^s \prod_{j=0}^{k-1} f_i^{(j)} \tau^k \in \mathbb{H}_s\{\{\tau\}\}.$$

Then:

$$\forall a \in \mathbb{A}_s, \quad \exp_{\varphi_s} a = \varphi_{s,a} \exp_{\varphi_s}.$$

Furthermore  $\exp_{\varphi_s}$  converges on  $\mathbb{H}_{s,\infty}$ .

**Proposition 5.1.** *Assume that  $s \equiv 1 \pmod{q-1}$ . The  $\mathbb{A}_s$ -module  $\text{Ker}(\exp_{\varphi_s} : \mathbb{H}_{s,\infty} \rightarrow \mathbb{H}_{s,\infty})$  is a finitely generated  $\mathbb{A}_s$ -module, discrete in  $\mathbb{H}_{s,\infty}$  and of rank  $|\text{Pic}(A)|$ . In particular,  $\text{Ker} \exp_{\varphi_s}$  is an  $\mathbb{A}_s$ -lattice in  $\{x \in \mathbb{H}_{s,\infty}, \forall a \in A \setminus \{0\}, \sigma_a(x) = \text{sgn}(a)q^{n(\phi)(s-1)}x\}$ . Furthermore, if  $s \not\equiv 1 \pmod{q-1}$ , then:*

$$\text{Ker} \exp_{\varphi_s} = \{0\}.$$

*Proof.* One can show that, for any  $s$ ,  $\text{Ker} \exp_{\varphi_s}$  is a finitely generated  $\mathbb{A}_s$ -module and is discrete in  $\mathbb{H}_{s,\infty}$ .

We view  $\mathbb{H}_s$  as a subfield of  $k_s[\mathbb{F}_\infty][\pi_*)$ . There exists  $\mathcal{G} \subset G$  a system of representatives of  $\frac{G}{\text{Gal}(H/H_A)}$ , such that:

$$\forall x \in \mathbb{H}_s, \quad \kappa(x) = (\sigma(x))_{\sigma \in \mathcal{G}}.$$

By Proposition 3.8, for  $i = 1, \dots, s$ ,  $\sigma \in \mathcal{G}$ , we can select a non-zero element  $U_{i,\sigma} \in L_s = k_s[\mathbb{F}_\infty][\pi_*)$  such that:

$$\tau(U_{i,\sigma}) = \sigma(f_i)U_{i,\sigma}.$$

Thus, by similar arguments to those of the proof of Lemma 3.5, we get:

$$\text{Ker} \exp_{\sigma(\varphi_s)}|_{L_s} = \frac{\Lambda(\phi^\sigma)k_s}{\prod_{i=1}^s U_{i,\sigma}}.$$

Recall that (see Proposition 3.8):

$$U_{i,\sigma} \in \Lambda(\phi^\sigma)k_s \subset (q^{d_\infty} \sqrt[q]{-\pi})^{-q^{n(\phi)}} \mathbb{K}_{s,\infty},$$

and (Lemma 3.4):

$$\Lambda(\phi^\sigma)k_s \subset (q^{d_\infty} \sqrt[q]{-\pi})^{-q^{n(\phi)}} \mathbb{K}_{s,\infty}.$$

Thus:

$$\text{Ker} \exp_{\sigma(\varphi_s)}|_{L_s} \subset (q^{d_\infty} \sqrt[q]{-\pi})^{q^{n(\phi)(s-1)}} \mathbb{K}_{s,\infty}.$$

Thus, if  $s \equiv 1 \pmod{q-1}$ , we get:

$$\text{Ker} \exp_{\sigma(\varphi_s)}|_{k_s[\mathbb{F}_\infty][\pi_*)} = \frac{\Lambda(\phi^\sigma)k_s}{\prod_{i=1}^s U_{i,\sigma}},$$

and if  $s \not\equiv 1 \pmod{q-1}$ :

$$\text{Ker} \exp_{\sigma(\varphi_s)}|_{k_s[\mathbb{F}_\infty][\pi_*)} = \{0\}.$$

□

**Remark 5.2.** Let  $\mathbb{H}'_s = \text{Frac}(H_A \otimes_{\mathbb{F}_q} k_s)$ . Let  $I = aA, a \in A \setminus \{0\}$ , and  $\sigma = \sigma_I \in \text{Gal}(H/H_A)$ . We have already noticed that:

$$\sigma(f) = \text{sgn}(a)q^{n(\phi) - q^{n(\phi)+1}} f.$$

We verify that:

$$\forall \sigma \in \text{Gal}(H/H_A), \quad \varphi_s^\sigma = \varphi_s \Leftrightarrow s \equiv 1 \pmod{\frac{q^{d_\infty} - 1}{q - 1}}.$$

In particular, when  $s \equiv 1 \pmod{q^{d_\infty} - 1}$ ,  $\varphi_s$  is defined over  $\mathbb{H}'_s$ ,  $\exp_{\varphi_s} : \mathbb{H}_s \rightarrow \mathbb{H}_s$  is  $\text{Gal}(H/H_A)$ -equivariant, and  $\text{Ker} \exp_{\varphi_s}$  is an  $\mathbb{A}_s$ -lattice in  $\mathbb{H}'_{s,\infty} := \mathbb{H}'_s \otimes_{\mathbb{K}_s} \mathbb{K}_{s,\infty}$ .

We introduce (see [4]):

$$\mathcal{L}_s = \sum_{I \in \mathcal{I}(A), I \subset A} \frac{\prod_{k=1}^s \rho_k(u_I)}{\psi_\phi(I)} \sigma_I \in \mathbb{H}_{s,\infty}[G]^\times.$$

**Theorem 5.3.** *Let  $s \equiv 1 \pmod{\frac{q^{d_\infty}-1}{q-1}}$ . Set:*

$$W'_s = (\oplus_{i_1, \dots, i_s \geq 0} B \prod_{k=1}^s f_k \cdots f_k^{(i_k-1)})^{\text{Gal}(H/H_A)}.$$

Then:

$$\exp_{\varphi_s}(\mathcal{L}_s W'_s) \subset W'_s.$$

*Proof.* By our assumption on  $s$ , and by Lemma 3.1, we get:

$$\mathcal{L}_s \in \mathbb{H}'_{s,\infty}[G]^\times.$$

The result is then a consequence of the above remark and [4], Corollary 4.10.  $\square$

**Remark 5.4.** Let  $W'_s = (\oplus_{i_1, \dots, i_s \geq 0} B \prod_{k=1}^s f_k \cdots f_k^{(i_k-1)})^{\text{Gal}(H/H_A)}$ . By Lemma 3.3, there exists  $u \in B^\times$  such that:

$$\frac{f}{u} \in \text{Frac}(H_A \otimes_{\mathbb{F}_q} A).$$

In particular:

$$B = B'[u],$$

where we recall that  $B'$  is the integral closure of  $A$  in  $H_A$ . Thus:

$$W'_s = \oplus_{i_1, \dots, i_s \geq 0} B' u^{-\sum_{k=1}^s \frac{q^{i_k}-1}{q-1}} \prod_{k=1}^s f_k \cdots f_k^{(i_k-1)}.$$

Let  $\mathbb{W}'_s$  be the  $k_s$ -vector space generated by  $W'_s$ . Then, by the proof of [4], Lemma 4.4,  $\mathbb{W}'_s$  is a fractional ideal of  $\mathbb{B}'_s := B'[k_s]$ , and therefore  $\mathbb{W}'_s$  is an  $\mathbb{A}_s$ -lattice in  $\mathbb{H}'_{s,\infty}$ .

**Proposition 5.5.** *Let  $s \equiv 1 \pmod{\frac{q^{d_\infty}-1}{q-1}}$ . We set:*

$$\mathbb{U}_s = \{x \in \mathbb{H}'_{s,\infty}, \exp_{\varphi_s}(x) \in \mathbb{W}'_s\}.$$

Then  $\mathbb{U}_s$  is an  $\mathbb{A}_s$ -lattice in  $\mathbb{H}'_{s,\infty}$  and:

$$\mathcal{L}_s \mathbb{W}'_s \subset \mathbb{U}_s.$$

If furthermore  $s \equiv 1 \pmod{q^{d_\infty}-1}$ , then  $\frac{\mathbb{U}_s}{\text{Ker exp}_{\varphi_s}}$  is a finite dimensional  $k_s$ -vector space. In particular, there exists  $a \in \mathbb{A}_s \setminus \{0\}$  such that:

$$a \mathcal{L}_s \mathbb{W}'_s \subset \text{Ker exp}_{\varphi_s}.$$

*Proof.* Since  $\mathbb{W}'_s$  is an  $\mathbb{A}_s$ -lattice in  $\mathbb{H}'_{s,\infty}$ , we deduce that  $\mathbb{U}_s$  is discrete in  $\mathbb{H}'_{s,\infty}$  and is a finitely generated  $\mathbb{A}_s$ -module. By Theorem 5.6, we have:

$$\mathcal{L}_s \mathbb{W}'_s \subset \mathbb{U}_s.$$

Let  $G' = \text{Gal}(H_A/K)$ , and let  $\text{res} : \mathbb{H}'_{s,\infty}[G] \rightarrow \mathbb{H}'_{s,\infty}[G']$  be the usual restriction map, then:

$$\text{res}(\mathcal{L}_s) \in \mathbb{H}'_{s,\infty}[G']^\times.$$

Therefore  $\mathcal{L}_s \mathbb{W}'_s$  is an  $\mathbb{A}_s$ -lattice in  $\mathbb{H}'_{s,\infty}$ . We conclude that  $\mathbb{U}_s$  is an  $\mathbb{A}_s$ -lattice in  $\mathbb{H}'_{s,\infty}$ .



If  $s \equiv 1 \pmod{q^{d_\infty} - 1}$ , then  $\text{Ker exp}_{\varphi_s}$  is a  $\mathbb{A}_s$ -lattice in  $\mathbb{H}'_{s,\infty}$  by Proposition 5.1. The proposition follows.  $\square$

**Theorem 5.6.** *Let  $s \equiv 1 \pmod{q^{d_\infty} - 1}$ . We work in  $L_s := k_s[\mathbb{F}_\infty][(\sqrt[q^{d_\infty}-1]{-\pi})]$ . There exist non-zero elements  $\omega_1, \dots, \omega_s \in \mathbb{T}_s := \mathcal{A}_s[\mathbb{F}_\infty][(\sqrt[q^{d_\infty}-1]{-\pi})]$  such that:*

$$\tau(\omega_i) = f_i \omega_i.$$

There also exists  $h \in B \setminus \{0\}$  such that:

$$\forall x \in \mathbb{W}'_s, \quad \frac{\mathcal{L}_s(x) \prod_{k=1}^s \omega_i}{\tilde{\pi}} \in h\mathbb{K}_s.$$

Furthermore, if  $\phi$  is standard, then  $h \in \mathbb{F}_\infty^\times$ .

*Proof.* By Proposition 3.8, we have:

$$f_1, \dots, f_s \in \mathbb{T}_s^\times.$$

By the same proposition, there exist  $\omega_1, \dots, \omega_s \in \mathbb{T}_s \setminus \{0\}$  such that:

$$\tau(\omega_i) = f_i \omega_i.$$

We deduce, by Lemma 3.4 and Lemma 3.5, that:

$$\text{Ker exp}_{\varphi_s} \mid_L = \frac{h\tilde{\pi}I\mathbb{A}_s}{\prod_{k=1}^s \omega_i},$$

where  $I$  is some fractional ideal of  $A$ ,  $h \in H^\times$ . Let  $x \in \mathbb{W}'_s$ , by Proposition 5.5, we get:

$$\frac{\mathcal{L}_s(x) \prod_{k=1}^s \omega_i}{\tilde{\pi}} \in h\mathbb{K}_s.$$

$\square$

We end this section with an application of the above Theorem. Let  $\phi \in \text{Drin}$  such that  $\phi$  is standard, i.e.

$$\text{Ker exp}_\phi = \tilde{\pi}A.$$

Let  $f \in \text{Sht}$  be the shtuka function associated to  $\phi$ .

**Theorem 5.7.** *Let  $n \geq 1$ ,  $n \equiv 0 \pmod{q^{d_\infty} - 1}$ . Then, there exists  $b \in B' \setminus \{0\}$  such that we have the following property in  $\mathbb{C}_\infty$ :*

$$\frac{\sum_I \frac{\sigma_I(b)}{\psi_\phi(I)^n}}{\tilde{\pi}^n} \in H_A^\times.$$

*Proof.* Write  $n = q^k - s$ ,  $k \equiv 0 \pmod{d_\infty}$ ,  $s \equiv 1 \pmod{q^{d_\infty} - 1}$ .

1) Observe that the map  $u$  extends naturally into a map  $u : \mathcal{I}(A) \rightarrow \mathbb{H}^\times$ , such that:

$$\forall x \in K^\times, \quad u_{xA} = \frac{\rho(x)}{\text{sgn}(x)},$$

$$\forall I, J \in \mathcal{I}(A), \quad u_{IJ} = \sigma_I(u_J)u_I.$$

By Lemma 3.1, we deduce that for all  $l \geq 0$ ,  $\frac{\tau^l(u_I)}{u_I}$  has no zero and no pole at  $\xi$ . For  $m \geq 1$ ,  $m \equiv 0 \pmod{d_\infty}$ , let  $\chi_m : \mathcal{I}_A \rightarrow H_A^\times$ , such that:

$$\forall I \in \mathcal{I}(A), \quad \chi_m(I) = \frac{\tau^m(u_I)}{u_I} \Big|_\xi.$$

We observe that:

$$\begin{aligned} \forall x \in K^\times, \quad \chi_m(xA) &= 1, \\ \forall I, J \in \mathcal{I}(A), \quad \chi_m(IJ) &= \sigma_I(\chi_m(J))\chi_m(I). \end{aligned}$$

In particular, there exists  $b_m \in B' \setminus \{0\}$  such that:

$$\forall I \in \mathcal{I}(A), \quad \chi_m(I) = \frac{\sigma_I(b_m)}{b_m}.$$

2) By Theorem 5.6, we have:

$$\frac{\mathcal{L}_s(1) \prod_{j=1}^s \omega_j}{\tilde{\pi}} \in \mathbb{K}_s.$$

We now apply  $\tau^k$  to the above rationality result. We get:

$$\frac{\prod_{j=1}^s (f_j \cdots f_j^{(k-1)} \omega_j) \tau^k(\mathcal{L}_s(1))}{\tilde{\pi} q^k} \in \mathbb{K}_s.$$

Let  $j \in \{1, \dots, s\}$ . Let  $\mathbb{H}_{s,j} = H(\rho_k(K), k = 1, \dots, s, k \neq j)$ . Let  $\xi_j$  be the place of  $\mathbb{H}_s/\mathbb{H}_{s,j}$  which corresponds to the kernel of the homomorphism of  $\mathbb{H}_{s,j}$ -algebras:  $\rho_j(A)[\mathbb{H}_{s,j}] \rightarrow \mathbb{H}_{s,j}, \rho_j(a) \mapsto a$ . By Proposition 3.11, there exists  $x_j \in K(\rho_j(K))^\times$  such that we have (recall that  $e_1(\phi) \neq 0$ ) :

$$x_j f_j \cdots f_j^{(k-1)} \omega_j \mid_{\xi_j} \in \tilde{\pi} H_A^\times.$$

Now:

$$\tau^k(\mathcal{L}_s(1)) = \sum_I \frac{\prod_{j=1}^s \rho_j(u_I)}{\psi_\phi(I) q^k} \prod_{j=1}^s \frac{\tau^k(\rho_j(u_I))}{\rho_j(u_I)}.$$

Therefore, there exists  $b \in B' \setminus \{0\}$  such that:

$$\tau^k(\mathcal{L}_s(1)) \mid_{\xi_1, \dots, \xi_s} = \frac{1}{b} \prod_P \left(1 - \frac{1}{\psi_\phi(P) q^{k-s}} (P, H/K)^{-1}(b)\right) \in K_\infty^\times.$$

The Theorem follows.  $\square$

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